A SYSTEM OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS.[1]

By G. N. BAUER and H. L. SLOBIN, University of Minnesota.

The equations under consideration are of the form

$$(1) \qquad P_1(x, y) = 0, \qquad P_2(x, y) = 0, \qquad T(x, y) = 0,$$

where $P_1(x, y)$ and $P_2(x, y)$ are polynomials in x and y with algebraic coefficients, and where $T(x, y) = 0$ denotes an equation which, in general, is not satisfied if both x and y are algebraic numbers. The equation $T(x, y) = 0$ may be satisfied by a finite number of pairs of algebraic numbers. Thus, the equation $x - e^y = 0$ is satisfied for $x = 1$, $y = 0$ and for no other pair of algebraic numbers. Other equations of this type are

$$P_1(x, y) + P_3(x, y) \tan P_2(x, y) = 0, \qquad P_1(x, y)e^{P_2(x, y)} + P_3(x, y) = 0,$$

where $P_1(x, y)$ and $P_3(x, y)$ have no common factor.

It is readily seen that there exists a non-enumerable set of systems of the type (1), for $T(x, y) = 0$ may be of the form $T(x, y) \equiv P_1(x, y) + t = 0$ where t is a transcendental number. This equation is not satisfied for a single pair of algebraic numbers.[2]

Throughout this paper $P_i(x, y)$ represents a polynomial in x and y with algebraic coefficients. The curve represented by an equation of the form $P_i(x, y) = 0$ will be called a P curve. The letter t always represents a transcendental number, and $p_i(t)$ represents a polynomial in t with algebraic coefficients. The curves represented by equations of the form $T(x, y) = 0$ are called T curves.

Before considering the system (1) attention is directed to a few preliminary considerations.

THEOREM 1. *The elimination of x or y from the equations*

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0,$$

produces a polynomial equation in one unknown, with algebraic coefficients.

The elimination of either x or y gives a polynomial equation whose coefficients are derived from the coefficients of the two given polynomials by the operations of addition and multiplication. Hence the theorem is evident.[3, 4]

THEOREM 2. *The equations*

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0,$$

are satisfied simultaneously by algebraic numbers only.

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[1] Read before the Chicago Section of the American Mathematical Society, December 22, 1916. *Bull. Am. Math. Soc.*, Vol. 23, p. 256.

[2] Bauer, G. N., and Slobin, H. L., "Some Transcendental Curves and Numbers," *Rendiconti del Circolo Matematico di Palermo*, Vol. 36, 1916, pp. 327–332.

[3] Burnside and Panton, *Theory of Equations*, p. 349.

[4] Bachmann, *Vorlesungen über die Natur der Zahlen*, 1892, p. 20.

This follows directly from theorem 1, for upon eliminating y, for example, the resulting equation can be satisfied by algebraic numbers only.[1]

THEOREM 3. *The system*

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0, \qquad T(x, y) = 0,$$

can have no simultaneous solution except possibly one or more of the pairs of algebraic numbers which satisfy $T(x, y) = 0$.

This is apparent since, by theorem 2, the equations $P_1(x, y) = 0$, $P_2(x, y) = 0$ can be satisfied only by pairs of algebraic numbers, while, in general, $T(x, y) = 0$ cannot be satisfied by such a pair of numbers.

Thus, the system

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0, \qquad y - \sin x = 0,$$

can have no simultaneous solution excepting possibly (0, 0).

Likewise, the system

$$P_1(x, y) = 0, \qquad P_2(x, y) = 0, \qquad P_3(x, y) - t = 0,$$

can have no simultaneous solution.

It is evident that no two P curves can intersect on a T curve unless, by way of exception, the isolated points on the curve represented by $T(x, y) = 0$ whose coördinates are both algebraic numbers, happen to be among the intersections of the two P curves.

If a P curve cut a T curve, no other P curve can pass through any transcendental point of intersection; hence any curve which passes through a transcendental point of intersection of a P curve and a T curve is not a P curve. It does not follow from this that the curve is a T curve since it has not been shown that the P and T curves exhaust all possibilities. In fact, the equation

$$P_1(x, y)P_2(x, y) + tP_1(x, y) = 0$$

represents a curve that is neither a P curve nor a T curve.

THEOREM 4. *The equation* $P_1(x, y) = 0$ *can be satisfied only by a pair of numbers both of which are algebraic or both of which are transcendental.*

Let x be an algebraic number, then $P_1(x, y) = 0$ may be considered a polynomial with algebraic coefficients; hence y must be an algebraic number. If on the other hand x is a transcendental number, y cannot be algebraic, otherwise a transcendental number would satisfy an algebraic equation with algebraic coefficients.

THEOREM 5. *If a P curve (not a straight line) intersect a T curve (not in its isolated algebraic points), then the slope of any line through any algebraic point of the plane and through any point of intersection is a transcendental number.* (Any point of the plane whose coördinates are algebraic numbers is called an algebraic point.)

[1] Bachmann, *ibid.*, p. 21.

Let A be any intersection of a P curve and a T curve. Its coördinates are both transcendental numbers (t_1, t_2) since by hypothesis they cannot both be algebraic, and by theorem 3 one cannot be algebraic while the other is transcendental. Let (a_1, a_2) be the coördinates of any algebraic point of the plane. Let us assume that the slope is algebraic; then the equation is of the form

$$y - a_2 = m(x - a_1),$$

where m is an algebraic number. Then we have three equations, two representing P curves and one representing a T curve, intersecting in a common point, both of whose coördinates are transcendental numbers. But this is impossible by theorem 3. Hence m must be a transcendental number.

If the P curve is a straight line the theorem is true for all algebraic points of the plane not on the line. The theorem obviously does not apply to the algebraic points on the line, if P be a line, since the equation of the P curve is an algebraic equation, and hence the slope is an algebraic number.

Also, if the given P curve is a line, and if the slope in question is an algebraic number, then any algebraic point through which the line may be drawn must lie on the given curve, *i. e.*, the line so constructed must coincide with the given P line.

Corollary 1. Any line with an algebraic slope, passing through the transcendental intersections of a P and a T curve, does not pass through any algebraic point.

If the line pass through an algebraic point, it would be possible to write the equation in terms of the coördinates of the point and the slope, and hence the line would be a P curve. Then two P curves and a T curve would intersect in a transcendental point, which is impossible.

Corollary 2. If the equation of a P curve is of the form

$$P_1(x, y) - kP_2(x, y) = 0$$

where k is any algebraic number, the multiple points of the P curve due to the intersection of $P_1(x, y) = 0$ and $P_2(x, y) = 0$ cannot be among the transcendental points on a T curve.

THEOREM 6. *If a P curve (not a circle) cut a T curve (not in its algebraic points), then the distance between a point of intersection and any algebraic point of the plane is a transcendental number.*

Let the coördinates of any algebraic point of the plane be given by (a_1, a_2), and let d be the distance from this point to a point of intersection of a P curve and a T curve. Let us assume that d is an algebraic number. Then, with (a_1, a_2) as a center pass a circle through the point of intersection. The radius of the circle is d. Also the equation of the circle is given in terms of a_1, a_2 and d, and hence it is a P curve. We then have two P curves intersecting a T curve in a transcendental point which, by theorem 3, is impossible. Hence d is not an algebraic number, and must therefore be transcendental.

If the given P curve is a circle, the theorem is still true for all algebraic points of the plane with the exception of the center of the given circle. The theorem evidently does not apply to the center of the circle.

Corollary. If a circle which is a P curve cut a T curve (not in its isolated algebraic points), then the center of the circle is the only algebraic point in the plane whose distance from a point of intersection is an algebraic number.

THEOREM 7. *No circle with an algebraic radius, r, whose center is an intersection of a P curve and a T curve, passes through an algebraic point of the plane* (where r is not equal to the radius of the P curve, in case P happens to be a circle).

For, if the circle passes through an algebraic point (a_1, a_2), it is possible to construct a circle with this point as a center, and r as a radius. It would therefore be a P curve, and two P curves would intersect a T curve, which is impossible.

THEOREM 8. *If a tangent or a normal to a P curve passes through an algebraic point (a_1, a_2), it cannot pass through the intersection of the P curve and a T curve, unless the intersection happens to be one of the finite number of isolated algebraic points of the T curve.*

The equations of the tangents and normals to the P curve, passing through the point (a_1, a_2) can be expressed in terms of a_1, a_2 and the partial derivatives of $P(x, y)$. Hence these equations represent P curves. But it is impossible for two P curves and a T curve to intersect; hence the theorem.

Corollary. A line tangent to a P curve at an algebraic point is a P curve.

THEOREM 9. *Any normal [tangent] line to a P curve, through any algebraic point (a_1, a_2) of the plane is normal [tangent] at an algebraic point.*

This follows directly from the fact that under the stated conditions, the tangent and normal are P curves.

THEOREM 10. *Any line drawn tangent [normal] to a P curve at a transcendental point is a T curve.*

For, if it contained an algebraic point, we could set up

$$(1) \qquad a_1 \frac{\partial P(x, y)}{\partial x} + a_2 \frac{\partial P(x, y)}{\partial y} = 0,$$

$$(2) \qquad (a_1 - x) \frac{\partial P(x, y)}{\partial y} - (a_2 - y) \frac{\partial P(x, y)}{\partial x} = 0,$$

respectively the curve passing through the points of contact of the tangents drawn from (a_1, a_2) to $P(x, y) = 0$, and the curve passing through the feet of all normals drawn from (a_1, a_2) to $P(x, y) = 0$, and find two P curves intersecting in a transcendental point. But this is impossible. Hence it is a T curve.

If the P curve is a straight line, the first member of (1) is *constant*, and the theorem does not apply. If the P curve is a circle and (a_1, a_2) the center of circle (2) is identically 0, and the theorem does not apply in that case.

THEOREM 11. *No transcendental point of a P curve is a singular point.*

For the singular points demand the simultaneous existence of

$$P(x, y) = 0, \qquad \frac{\partial P(x, y)}{\partial x} = 0, \qquad \frac{\partial P(x, y)}{\partial y} = 0,$$

which represent P curves and can be satisfied simultaneously only by algebraic

numbers. If one of the derivatives reduces identically to 0, the remaining equation must be satisfied simultaneously with the given equation $P(x, y) = 0$. This demands that the solution be algebraic numbers.

Corollary. A T curve cannot intersect a P curve in the singular points of the P curve, unless the singular point happens to be one of a finite number of algebraic points of the T curve.

THEOREM 12. *All of the singular points of*

$$R(x, y) \equiv P_1(x, y) + p(t)P_2(x, y) = 0, \qquad [1]$$

which require

$$\frac{\partial R(x, y)}{\partial x} = 0 \qquad [2]$$

and

$$\frac{\partial R(x, y)}{\partial y} = 0, \qquad [3]$$

are among the intersections of $P_1(x, y) = 0$ and $P_2(x, y) = 0$, if there are any at all.

The equations [2] and [3], in general, contain t. Eliminating t between [1] and [2] and also between [2] and [3] we have two equations representing P curves, which, taken together with [1] give a system which cannot be satisfied simultaneously excepting for the values common to $P_1(x, y) = 0$ and $P_2(x, y) = 0$.

Corollary. The curves represented by the equations of the type $P_1(x, y) + p(t) = 0$ have no singular points.

THEOREM 13. *All the singularities of*

$$R(x, y) \equiv p_1(t)P_1(x, y) + p_2(t)P_2(x, y) = 0$$

are found among the intersections of $P_1(x, y) = 0$ and $P_2(x, y) = 0$, if any exist, provided $p_1(t) \not\equiv \lambda p_2(t)$.

By setting up the partial derivatives of $R(x, y)$ with respect to x and y, the argument is seen to follow the lines used in the last demonstration.

THEOREM 14. *The singularities of*

$$[1] \qquad R(x, y) \equiv p_0(t)P_0(x, y) + p_1(t)P_1(x, y) + \cdots + p_n(t)P_n(x, y) = 0,$$

if any exist, are among the simultaneous solutions of

$$P_0(x, y) = 0, \qquad P_1(x, y) = 0, \qquad \cdots, \qquad P_n(x, y) = 0$$

provided

$$[2] \qquad \| a_0, b_1, c_2, \cdots, k_n \| \not\equiv 0$$

where the elements entering the matrix are the coefficients of the polynomials $p_0(t)$, $p_1(t)$, $\cdots$, $p_n(t)$.

Let us write

$$p_i(t) = a_i + b_it + c_it^2 + \cdots \quad [i = 0, 1, 2, \cdots n]$$

and let the highest power of t occurring in any polynomial be designated by j.

Also, representing $P_i(x, y)$ by A_i it is seen that the following $j + 1$ equations must subsist in order that there may be a solution:

$$\begin{aligned} a_0A_0 + a_1A_1 + \cdots + a_nA_n &= 0, \\ b_0A_0 + b_1A_1 + \cdots + b_nA_n &= 0, \\ &\vdots \\ k_0A_0 + k_1A_1 + \cdots + k_nA_n &= 0; \end{aligned}$$

for the coefficients of the individual powers of t must reduce to 0, since by placing

$$\frac{\partial R(x, y)}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial R(x, y)}{\partial y} = 0$$

it is seen that the singular points must be algebraic points. This requires that the matrix [2] be equal to 0 or that $A_0 = A_1 = \cdots = A_n = 0$. But by hypothesis the matrix is not equal to 0 and hence $A_0, A_1, \cdots, A_n$ must reduce to 0, i. e., the singularities are among the simultaneous solutions of $P_0 = 0$, $P_1 = 0$, $\cdots$, $P_n = 0$.

Corollary. The total number of singular points, which require

$$R(x, y) = 0, \qquad \frac{\partial R(x, y)}{\partial x} = 0, \qquad \frac{\partial R(x, y)}{\partial y} = 0,$$

cannot exceed jk where j and k are the degrees of the two polynomials of lowest degree, which enter in [1].

It is evident that many such equations [1] may occur where jk would be much less than the maximum number of such points possible as determined from the table of Plückerian Characteristics. Thus for example

$$p_1(t)x + p_2(t)y + p_3(t)P_n(x, y) = 0$$

where $P_n(x, y)$ is a polynomial of the nth degree with the absolute constant not 0, has no such singularities, for $x = 0$, $y = 0$ do not satisfy $P_n(x, y) = 0$.

The various theorems pertaining to singularities may easily be extended to geometry of higher dimensions.